plied by 100 L/r_p. The resulting currents exhibit the expected dependence on the Knudsen number but are large by a factor of about two. It is believed that this is an end effect, since the end-on cross-sectional area of the sheath for the experimental probes was of the same order as the circumferential area.

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Implicit Rigid Body Motion in Curved Finite Elements

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It is generally conceded that rigid body motion must be adequately represented in curved as well as flat finite elements. However, there is still much disagreement regarding whether the rigid body motion should be represented explicitly or whether an implicit representation is sufficient. This Note pertains to the implicit representation where the rigid body modes are resurrected with mesh refinement.

Stricklin, Navaratna and Pian[1] ignored the explicit representation of rigid body motion in the development of a curved element for a shell of revolution. The authors used a cubic function in the meridional distance for the normal displacement and linear functions for the displacements in the meridional and circumferential directions. They simply stated that the explicit inclusion of rigid body motion was not necessary. It was later shown by Haisler and Stricklin[2] that rigid body motion is, in fact, recovered with mesh refinement. In certain cases the recovery may however be too slow to be of practical value. Gallagher[3] and Schmit, Bogner, and Fox[4] later developed curved cylindrical elements for the analysis of shells by the direct stiffness method. In Ref. 4 it was found that linear displacement functions were insufficient for the adequate representation of rigid body motion and consequently, bicubic functions were used for all displacements. More recently many authors have developed finite elements based on the implicit inclusion of rigid body motion. References 5 and 6 present reviews of these elements.

The curved shell of revolution element presented in Ref. 1 has been used in computer codes developed at Texas A&M University[7,8] and at M.I.T.[9,10] Experience has shown that rigid body motion is represented with element refinement. However, recent experience with shells which circumscribe very large angles (>90°) has demonstrated that a large number of elements (>50) may be required for an adequate solution. This offers no difficulty for linear analyses but requires the expenditure of large amounts of computer time for nonlinear analyses. Consequently, a better element was developed and is reported in the present Note.

The curvature of the shell is represented by the same procedure as given in Ref. 1. However, in the present research the displacements are given by

$$
\begin{aligned}
w &= \Sigma(\alpha_1{}^i + \alpha_2{}^i S + \alpha_3{}^i S^2 + \alpha_4{}^i S^3)\cos(i\theta)\\
u &= \Sigma[\alpha_5{}^i + \alpha_6{}^i S + \beta_1{}^i S(S-L) + \beta_2{}^i S^2(S-L)]\cos(i\theta) \qquad (1)\\
v &= \Sigma[\alpha_7{}^i + \alpha_8{}^i S + \beta_3{}^i S(S-L) + \beta_4{}^i S^2(S-L)]\sin(i\theta)
\end{aligned}
$$

where w, u, and v are the displacements in the normal, meridional, and circumferential directions, respectively, and L is the length of the element in the meridional direction. S is the meridional distance along the element.

Substituting Eqs. (1) into the strain energy expression, the internal energy may be written as

$$U = \frac{1}{2}[\alpha \quad \beta]\begin{bmatrix} L_{\alpha\alpha} & L_{\alpha\beta} \\ L_{\beta\alpha} & L_{\beta\beta}\end{bmatrix}\begin{Bmatrix}\alpha \\ \beta\end{Bmatrix} \qquad (2)$$

Since the coefficients of the β terms vanish at both ends of the elements, the β terms may be eliminated by static condensation. This yields an element stiffness matrix in terms of the α coefficients.

$$[L] = [L_{\alpha\alpha}] - [L_{\alpha\beta}][L_{\beta\beta}]^{-1}[L_{\beta\alpha}] \qquad (3)$$

The transformation to global directions follows the same procedure as given in Ref. 7.

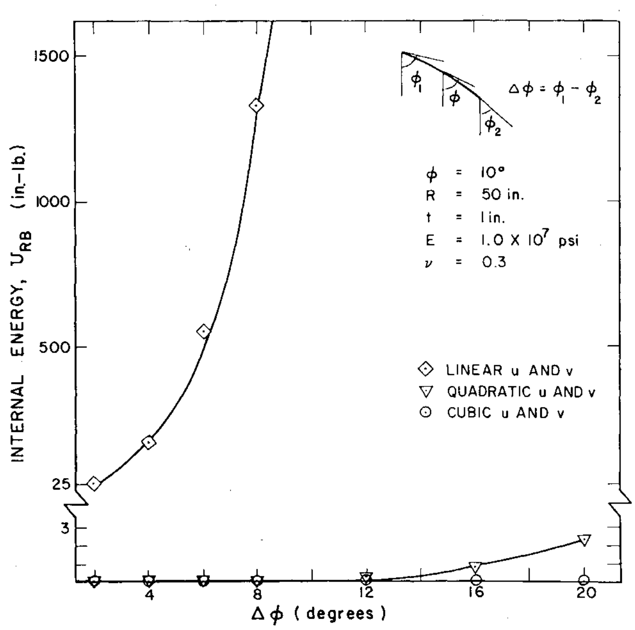

Fig. 1 Internal energy associated with an axial rigid body motion of a hemispheric element located near the base.

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Table 1 Curved beam element rotation mode eigenvalues[a]

$\Delta\phi$	Linear	Quadratic	Cubic
2	2013.	0.05688	0.00005
4	7006.	0.00234	0.00002
6	12177.	0.02387	0.00019
8	17179.	0.04258	0.00079
12	26794.	0.21930	0.00618
16	35933.	0.88204	0.02636
20	44388.	2.46182	0.08052

[a] $E = 1.0 \times 10^7$; $\nu = 0.3$; $t = 2$ in.; width $= 2$ in.; $I = 1.33$ in.^{4}.

Examination of the displacement function given by Eq. (1) reveals that there is one rigid body mode associated with the zeroth harmonic and two associated with the first harmonic. Eigenvalue analyses were conducted on the matrix $[L]$ for both harmonics. It was found that the representation of rigid body translation in the zeroth harmonic yielded the most critical case. For this case a plot of the internal energy versus change in angle is given in Fig. 1. The quadratic representation was obtained by setting $\beta_2 = \beta_4 = 0$. It is noted that quadratic and cubic displacement functions show a tremendous improvement over the results obtained for a linear u and v. A cubic displacement function in w was used in all three cases.

The results for a hemispherical shell undergoing large rigid body translation are shown in Fig. 2. This example confirms the eigenvalue analysis that the quadratic and cubic displacement functions are considerably better than the linear functions.

An eigenvalue analysis was also conducted on a curved beam element. A cubic function was used for the normal displacement and linear, quadratic and cubic functions were used for the meridional displacements. The results for the largest eigenvalue which should be zero for rigid body motion are shown in Table 1.

It is again noted that there is a tremendous improvement in going from a linear to a quadratic function in the meridional displacement. For both the shell and beam element the lowest elastic eigenvalue was of the order of 1.0×10^6.

In summary it has been demonstrated in this Note that implicit rigid body motion is much better represented when going from a linear to a quadratic displacement function. Static condensation makes it possible to use the higher order displacement function without increasing the size of the resulting element stiffness matrix.

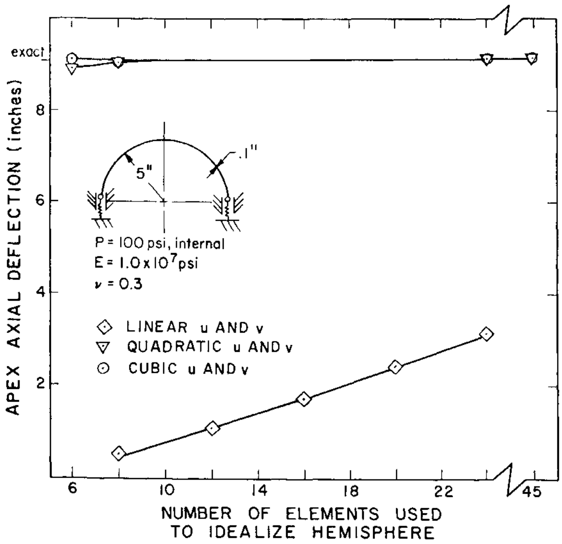

Fig. 2 Deflection analysis of a hemisphere with large rigid body motion.

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Static Pressure Measurements near an Oblique Shock Wave

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Nomenclature

d = probe diameter
l = probe length
M = Mach number
p = static pressure
p_t' = pitot pressure
p_{to} = reservoir pressure

Introduction

THIS Note is concerned with appraising readings of relatively short static pressure probes in the vicinity of an oblique shock wave. Such probes are used along with pitot tubes to determine the Mach number distribution in supersonic flowfields. In the absence of shock waves in the flow,

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